Example Solutions

Example: show that the derivative of $y = x^n$ is nx^{n-1}

Using the definition of the derivative,

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

expanding binomially

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{\left(x^n + nhx^{n-1} + \frac{n(n-1)}{2} h^2 x^{n-2} + \ldots + h^n \right) - x^n}{h}$$

to first order $= \lim_{h \to 0} \frac{nhx^{n-1}}{h}$

$$= nx^{n-1}$$

Example: If $x = \rho\cos\phi$ and $y = \rho\sin\phi$, then does $\dfrac{\partial\rho}{\partial x} = \dfrac{1}{\dfrac{\partial x}{\partial\rho}}$?

Differentiating,

$$\frac{\partial x}{\partial\rho} = \cos\phi .$$

But,

$$\rho = \left(x^2 + y^2\right)^{\frac{1}{2}},\ \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

so

$$\frac{\partial\rho}{\partial x} = \frac{x}{\left(x^2 + y^2\right)^{\frac{1}{2}}} = \cos\phi$$

It appears that $\dfrac{\partial\rho}{\partial x} = \dfrac{\partial x}{\partial\rho}$!

But, different variables were kept constant during the differentiation! We have actually shown that

$$\left(\frac{\partial x}{\partial\rho}\right)_\phi = \left(\frac{\partial\rho}{\partial x}\right)_y .$$

Since $\rho = x\sec\phi$ then $\left(\dfrac{\partial\rho}{\partial x}\right)_\phi = \sec\phi$ and so

$$\left(\frac{\partial\rho}{\partial x}\right)_\phi = \frac{1}{\left(\dfrac{\partial x}{\partial\rho}\right)_\phi} .$$

Example: If $f = xy$ calculate the partial derivatives with respect to the polar coordinates ρ and ϕ, by the chain rule and explicitly by substitution.

Chain rule: $\dfrac{\partial f}{\partial \rho} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial \rho} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial \rho}$,

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \phi}$$

so
$$\frac{\partial f}{\partial \rho} = (y)(\cos\phi) + (x)(\sin\phi) = 2\rho\sin\phi\cos\phi$$
$$= \rho\sin(2\phi)$$

$$\frac{\partial f}{\partial \phi} = (y)(-\rho\sin\phi) + (x)(\rho\cos\phi) = \rho^2\left(\cos^2\phi - \sin^2\phi\right)$$
$$= \rho^2\cos 2\phi$$

Explicitly: $f = (\rho\cos\phi)(\rho\sin\phi) = \dfrac{1}{2}\rho^2\sin 2\phi$

and we obtain $\dfrac{\partial f}{\partial \rho}$ and $\dfrac{\partial f}{\partial \phi}$ by inspection.

Example: Transform the operator $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ into polar coordinates.

We begin by evaluating the first derivatives by means of the chain rule.

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial x}=\frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r}-\frac{y}{x^2+y^2}\frac{\partial}{\partial \phi}$$

$$=\cos\phi\frac{\partial}{\partial r}-\frac{\sin\phi}{r}\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial r}+\frac{x}{x^2+y^2}\frac{\partial}{\partial \phi}$$

$$=\sin\phi\frac{\partial}{\partial r}+\frac{\cos\phi}{r}\frac{\partial}{\partial \phi}$$

and then applying the operators to themselves

$$\frac{\partial^2}{\partial x^2}=\cos^2\phi\frac{\partial^2}{\partial r^2}+\frac{\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial \phi}-\frac{\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial \phi}$$

$$+\frac{\sin^2\phi}{r}\frac{\partial}{\partial r}-\frac{\sin\phi\cos\phi}{r}\frac{\partial}{\partial \phi\partial r}+\frac{\sin\phi\cos\phi}{r}\frac{\partial}{\partial \phi}+\frac{\sin^2\phi}{r^2}\frac{\partial^2}{\partial \phi^2}$$

$$\frac{\partial^2}{\partial y^2}=\sin^2\phi\frac{\partial^2}{\partial r^2}-\frac{\sin\phi\cos\phi}{r^2}\frac{\partial}{\partial \phi}+\frac{\sin\phi\cos\phi}{r}\frac{\partial^2}{\partial r\partial \phi}$$

$$+\frac{\cos^2\phi}{r}\frac{\partial}{\partial r}+\frac{\sin\phi\cos\phi}{r}\frac{\partial}{\partial \phi\partial r}-\frac{\sin\phi\cos\phi}{r}\frac{\partial}{\partial \phi}+\frac{\cos^2\phi}{r^2}\frac{\partial^2}{\partial \phi^2}$$

Hence $$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial}{\partial \varphi^2}$$