PROBLEMS

3 Approximate Methods

3.1 Expand the *error function* $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$

in a Taylor series about $x = 0$.

3.2 Show that for large values of x

$$\operatorname{erf}(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}} \left[\frac{1}{x} - \frac{1}{2x^3} + \frac{1 \cdot 3}{2^2 x^5} - \frac{1 \cdot 3 \cdot 5}{2^3 x^7} + \cdots \right]$$

3.3 Obtain the asymptotic expansion of the

exponential integral $Ei(x) = \int_x^\infty \frac{e^{-u}}{u} \mathrm{d}u$.

3.4 Use the *method of steepest descent* to show that the first term of the asymptotic expansion of $n!$ for $n >> 1$ is given by Stirling's formula:

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e} \right)^n.$$

3.5 Use the WKB method to find the energy levels of a particle confined to the one-dimensional potential $V(x) = m\omega^2 x^2 / 2$.

3.6 Use the WKB method to find the energy levels of a particle confined to the one-dimensional potential $V(x) = F|x|$.

3.7 Use the WKB method to find the energy levels of a particle confined to the one-dimensional potential $V(x) = \beta x^4$.

3.8 Find the energies of the highly excited states of a particle confined to the one-dimensional Coulomb potential $V(x) = -\alpha / |x|$.

3.9 Estimate the ground-state energy of the one-dimensional harmonic oscillator with potential energy $V(x) = m\omega^2 x^2 / 2$ using four variational wave functions:

(a) $\Psi(x) = A\exp(-\alpha|x|)$,

(b) $\Psi(x) = \begin{cases} A(1-|x|/a), \text{ for } |x| < a \\ 0, \text{ for } |x| \geq a \end{cases}$,

(c) $\Psi(x) = \begin{cases} A(1-x^2/a^2), \text{ for } |x| < a \\ 0, \text{ for } |x| \geq a \end{cases}$,

(d) $\Psi(x) = \dfrac{A}{x^2 + a^2}$.

Which of the four functions is a better approximation to the exact ground-state wave function of the harmonic oscillator?

3.10 Estimate the ground-state energy of the particle confined in the triangular quantum well potential $V(x) = \begin{cases} +\infty, & \text{for } x \leq 0 \\ Fx, & \text{for } x > 0 \end{cases}$, using the variational wave function

$$\Psi(x) = \begin{cases} 0, \text{ for } x \leq 0 \\ Axe^{-bx}, \text{ for } x > 0 \end{cases}.$$