PHYSICS EXAMINATION PROBLEMS
SOLUTIONS AND HINTS FOR STUDENT SELF-STUDY

Module Code	**PHY3140**
Name of module	**Methods of theoretical physics**
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1. $\Gamma(t)=\int\limits_0^{+\infty} x^{t-1}e^{-x}\,\mathrm{d}x \Rightarrow \Gamma(t+1)=\int\limits_0^{+\infty} x^{t}e^{-x}\,\mathrm{d}x = -x^t e^{-x}\Big|_0^\infty + t\int\limits_0^{+\infty} x^{t-1}e^{-x}\,\mathrm{d}x = t\Gamma(t)$. $n! = \Gamma(n+1)$.

$$\int\limits_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \left(\int\limits_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\int\limits_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y\right)^{1/2} = \left(\int\limits_0^{2\pi}\mathrm{d}\varphi\int\limits_0^{\infty} e^{-\rho^2}\rho\,\mathrm{d}\rho\right)^{1/2} = \left(\pi\int\limits_0^{\infty} e^{-t}\mathrm{d}t\right)^{1/2} = \sqrt{\pi}\,.$$

$$\Gamma\left(\frac{1}{2}\right) = \int\limits_{-\infty}^{\infty} x^{-1/2}e^{-x}\,\mathrm{d}x = 2\int\limits_0^{\infty} e^{-u^2}\,\mathrm{d}u = \sqrt{\pi}\,.$$

$$n! = \Gamma(n+1) = \int\limits_0^{+\infty} x^n e^{-x}\,\mathrm{d}x = \int\limits_0^{+\infty} e^{n\ln x - x}\,\mathrm{d}x = \left(\frac{n}{e}\right)^n \int\limits_{-n}^{+\infty} e^{n\ln\left(1+\frac{y}{n}\right)-y}\,\mathrm{d}y \xrightarrow[\text{(for } n>>1)]{}$$

$$\to \left(\frac{n}{e}\right)^n \int\limits_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2n}\right)\mathrm{d}y = \left(\frac{n}{e}\right)^n\sqrt{2\pi n} - \text{Stirling formula.}$$

2. $|A|^2 = \dfrac{2b^3}{\pi}$. Kinetic energy: $\langle T\rangle = \dfrac{\hbar^2}{4mb^2}$. Potential energy: $\langle V\rangle = \dfrac{m\omega^2 b^2}{2}$. $\langle E_{\min}\rangle = \sqrt{2}\,\dfrac{\hbar\omega}{2}$.
[Hint: Use contour integration.]

3. (i) (a) $P_2(x) = x\left[1-(1-x)^3\right]$; (b) $P_2(x) = x\left[1-(1-x)^6\right]$; (c) $P_2(x) = x\left[1-(1-x)^d\right]$.

(ii) (a) $\displaystyle\int\limits_0^{+\infty}\frac{\cos(t/\tau)}{t^2+1}\mathrm{d}t = \frac{1}{2}\mathrm{Re}\left[\int\limits_{-\infty}^{+\infty}\frac{\exp(ix/|\tau|)}{t^2+1}\mathrm{d}x\right] = \frac{\pi}{2}\exp\left(-1/|\tau|\right)$. [Hint: Use contour integration.]

(b) $\displaystyle\int\limits_0^{\infty}\exp\left(-x^{0.2}\right)\mathrm{d}x = 5\Gamma(5) = 5! = 120$. [Hint: Substitute $x = t^5$.]

4. (i) WKB $\Rightarrow$ for highly excited states $\displaystyle\int\limits_a^b\sqrt{2m[E-V(x)]}\,\mathrm{d}x = \pi\hbar\left(n+\frac{1}{2}\right)$, where a and b are the classical turning points. For $V = \dfrac{kx^2}{2}$, $E = \dfrac{\pi}{4\gamma}\hbar\left(\dfrac{k}{m}\right)^{1/2}\left(n+\dfrac{1}{2}\right)$, where $\gamma = \displaystyle\int\limits_0^1\sqrt{1-t^2}\,\mathrm{d}t = \frac{\pi}{4}$.

[Hint: Substitute $t = \sin\theta$.] Finally, $E = \hbar\left(\dfrac{k}{m}\right)^{1/2}\left(n+\dfrac{1}{2}\right) = \hbar\omega\left(n+\dfrac{1}{2}\right)$.

(ii) $u(x,y) = x^2 - y^2 + C$. Thus, $f(z) = z^2 + C$.

(iii) $\displaystyle\int\limits_0^{2\pi}\frac{\mathrm{d}\theta}{a+\cos\theta} = \frac{2\pi}{\sqrt{a^2-1}}$. [Hint: Substitute $z = \exp(i\theta)$ and use contour integration.]