Functions of a Complex Variable

FUNCTIONS

If to each of a set of complex numbers which a variable z may assume there corresponds one or more values of a variable w, then w is called ***a function of the complex variable*** z, written $w = f(z)$.

A function is *single-valued* if for each value of z there is only one value of w; otherwise it is ***multiple-valued*** or ***many-valued***. In general we can write $w = f(z) = u(x, y) + iv(x, y)$, where u and v are real functions of x and y.

Example 1:

$w = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$ so that $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$. These are called ***real*** and ***imaginary parts*** of $w = z^2$ respectively.

Unless otherwise specified we shall assume that $f(z)$ is single-valued. A function, which is multiple-valued, can be considered as a collection of single-valued functions.

LIMITS AND CONTINUITY

Definitions of limits and continuity for functions of a complex variable are similar to those for a real variable. Thus, $f(z)$ is said to have the ***limit*** l as z approaches z_0, if given any $\varepsilon > 0$, there exist a $\delta > 0$, such that $|f(z) - l| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

Similarly, $f(z)$ is said to be ***continuous*** at z_0 if, given any $\varepsilon > 0$, there exist a $\delta > 0$, such that $|f(z) - f(z_0)| < \varepsilon$ when $|z - z_0| < \delta$. Alternatively, $f(z)$ is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

Theorems on limits

If $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$, then

1. $\lim_{z \to z_0} \{f(z) + g(z)\} = A + B$
2. $\lim_{z \to z_0} \{f(z) - g(z)\} = A - B$
3. $\lim_{z \to z_0} \{f(z)g(z)\} = AB$
4. $\lim_{z \to z_0} \{f(z)/g(z)\} = A/B$ if $B \neq 0$

DERIVATIVES

If $f(z)$ is single-valued in some region of the z plane the ***derivative*** of $f(z)$, denoted by $f'(z)$, is defined as

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} \tag{1}$$

provided the limit exists independent of the manner in which $\Delta z \to 0$. If this limit exists for $z = z_0$, then $f(z)$ is called ***analytic*** at z_0. If the limit (1) exists for all z in a region ***R***, then $f(z)$ is called ***analytic in R***. In order to be analytic $f(z)$ must be single-valued and continuous. The converse, however, is not necessarily true.

We define elementary functions of a complex variable by a natural extension of the corresponding functions of a real variable. Where series expansions for real functions $f(x)$ exist, we can use as definitions the series with x replaced by z.

Example 2: We define

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots,$$

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots,$$

$$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots.$$

From these we can show that $e^z = e^{x+iy} = e^x(\cos y + i \sin y)$, as well as many other relations.

Example 3:

We define a^b as $e^{b \ln a}$ even when a and b are complex numbers. Since $e^{2k\pi i} = 1$, it follows that $e^{i\phi} = e^{i(\phi+2k\pi)}$ and we define

$$\ln z = \ln\left(\rho e^{i\phi}\right) = \ln \rho + i(\phi + 2k\pi).$$

Thus $\ln z$ is a many-valued function. The various single-valued functions of which this many-valued function is composed are called its ***branches***.

Rules for differentiating functions of a complex variable are much the same as for those of real variables. Thus

$$\frac{d}{dz}\left(z^n\right) = nz^{n-1},$$

$$\frac{d}{dz}\left(e^z\right) = e^z,$$

$$\frac{d}{dz}(\sin z) = \cos z,$$

$$\frac{d}{dz}(\cos z) = -\sin z,$$

etc.

CAUCHY-RIEMANN EQUATIONS

A necessary condition, that $w = f(z) = u(x,y) + iv(x,y)$ be analytic in a region ***R*** is that u and v satisfy the ***Cauchy-Riemann equations***

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2}$$

This condition is easy to prove first choosing $\Delta y = 0$ (thus $\Delta z = \Delta x$), then choosing $\Delta x = 0$ (thus $\Delta z = i\Delta y$), and finally equating the expressions for the derivatives of $f(z)$ obtained in these two cases. If the partial derivatives in (2) are continuous in ***R***, the equations are sufficient conditions that $f(z)$ be analytic in ***R***.

If the second derivatives of u and v with respect to x and y exist and are continuous, we find by differentiating (2) that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{3}$$

Thus u and v satisfy Laplace's equation in 2 dimensions. Functions satisfying Laplace's equation are called ***harmonic functions***.

INTEGRALS

If $f(z)$ is single-valued and continuous in a region ***R*** we define the ***integral*** of $f(z)$ along some path C in ***R*** from point z_1 to point z_2, where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, as

$$\int_C f(z)dz = \int_{(x_1+iy_1)}^{(x_2+iy_2)} (u + iv)(dx + idy) =$$

$$\int_{(x_1+iy_1)}^{(x_2+iy_2)} udx - vdy + i\int_{(x_1+iy_1)}^{(x_2+iy_2)} vdx + udy \qquad (4)$$

with this definition the integral of a function of a complex variable can be made to depend on line integrals for real functions already considered in Math-2 (see, e.g. Chapter 6 of Spiegel's textbook). An alternative definition based on the limit of a sum, as for functions of a real variable, can also be formulated and turns out to be equivalent to the one above.

The rules for complex integration are similar to those for real integrals. An important result is

$$\left|\int_C f(z)dz\right| \le \int_C |f(z)||dz| \le M\int_C ds = ML.$$

Here M is the upper bound of $|f(z)|$ on C, i.e. $|f(z)| \le M$, and L is the length of the path C.

CAUCHY'S THEOREM

Let C be a simple closed curve. If $f(z)$ is analytic within the region bounded by C as well as on C, then we have ***Cauchy's theorem*** that

$$\int_C f(z)dz = \oint_C f(z)dz = 0, \qquad (5)$$

where the second integral emphasises the fact that C is a simple closed curve.

Expressed in another way, equation (5) is equivalent to the statement that $\int_{z_1}^{z_2} f(z)dz$ has a value ***independent of the path*** joining z_1 and z_2. Such integrals can be evaluated as $F(z_2) - F(z_1)$ where $F'(z) = f(z)$.

The proof of Cauchy's theorem follows immediately from Eq. (4), Cauchy-Riemann equations (3) and Green's theorem in the plane (see Math-2 or Chapter 6 of Spiegel):

$$\oint_C (P\,dx + Q\,dy) = \iint_{\boldsymbol{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

Example 4: Since $f(z) = z$ is analytic everywhere

$$\oint_C z\,dz = 0. \quad \text{Also,} \quad \int_{2i}^{1+i} z\,dz = \left.\frac{z^2}{2}\right|_{2i}^{1+i} = i + 2$$

CAUCHY'S INTEGRAL FORMULAS

If $f(z)$ is analytic within and on a simple closed curve C and a is any point interior to C, then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a}\,dz, \qquad (6)$$

where C is traversed in the positive (counter-clockwise) sense.

Also, the nth derivative of $f(z)$ at $z = a$ is given by

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - a)^{n+1}}\,dz. \qquad (7)$$

Equations (6) and (7) are called ***Cauchy's integral formulas***. They are quite remarkable because they show that if the function $f(z)$ is known ***on*** the closed curve C then it is also known ***within*** C, and the various derivatives at points within C can be calculated. Thus if a function of a complex variable has a first derivative, it has all higher derivatives as well. This of course is not necessarily true for functions of real variables.

TAYLOR'S SERIES

Let $f(z)$ be analytic inside and on a circle having its centre at $z = a$. Then for all points z in the circle we have the ***Taylor series*** representation of $f(z)$ given by

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots \quad (8)$$

SINGULAR POINTS

A singular point of a function $f(z)$ is a value of z at which $f(z)$ fails to be analytic. If $f(z)$ is analytic everywhere in some region except at interior except at an interior point $z = a$, we call $z = a$ an ***isolated singularity*** of $f(z)$.

Example 5: If $f(z) = \dfrac{1}{(z-5)^3}$, then $z = 5$ is an isolated singularity of $f(z)$.

POLES

If $f(z)=\dfrac{\phi(z)}{(z-a)^n}$, $\phi(a)\neq 0$, where $\phi(z)$ is analytic everywhere in the region including $z=a$, and if n is a positive integer, then $f(z)$ has an isolated singularity at $z=a$ which is called a ***pole of order*** n. If $n=1$, the pole is often called a ***simple pole***; if $n=2$ it is called a ***double pole***, etc.

Example 6: $f(z)=\dfrac{z^3}{(z-5)^2(z+2)}$

has two singularities: a pole of the order 2 or double pole at $z=5$, and a pole of order 1 or simple pole at $z=-2$.

A function can have other types of singularities besides poles. For example $f(z)=\sqrt{z}$ has a ***branch point*** at $z=0$. The function $f(z)=\dfrac{\sin z}{z}$ has a singularity at $z=0$. However, since $\lim\limits_{z\to 0}\dfrac{\sin z}{z}$ is finite, we call such a singularity ***removable singularity***.

LAURENT SERIES

If $f(z)$ has a pole of order n at $z=a$ but is analytic at every other point inside and on a circle C with centre at a, then $(z-a)^n f(z)$ is analytic at all points inside and on C and has a Taylor series about $z=a$ so that

$$f(z)=\frac{a_{-n}}{(z-a)^n}+\frac{a_{-n+1}}{(z-a)^{n-1}}+\cdots+\frac{a_{-1}}{z-a}+ \qquad (9)$$

$$+a_0+a_1(z-a)+a_2(z-a)^2+\cdots$$

This is called a ***Laurent series*** for $f(z)$. The part $a_0+a_1(z-a)+a_2(z-a)^2+\cdots$ is called the ***analytic part***, while the remainder consisting of inverse powers of $z-a$ is called the ***principal part***. More generally, we refer to the series $\sum_{k=-\infty}^{\infty} a_k(z-a)^k$ as a Laurent series where the terms with $k<0$ constitute the principal part. A function which is analytic in a region bounded by two concentric circles having centre at $z=a$ can always be expanded in such a Laurent series.

It is possible to define various types of singularities of a function $f(z)$ from its Laurent series. For example, when the principal part of a Laurent series has a finite number of terms and $a_{-n} \neq 0$ while a_{-n-1}, a_{-n-2}, … are all zero, then $z = a$ is a pole of order n. If the principal part has infinitely many terms, $z = a$ is called an ***essential singularity*** or sometimes a ***pole of infinite order***.

Example 7:

The function $\exp(1/z) = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$ has an essential singularity at $z = 0$.

RESIDUES

The coefficients in (9) can be obtained by writing the coefficients for the Taylor series corresponding to $(z-a)^n f(z)$. The coefficient a_{-1}, called the residue of $f(z)$ at the pole $z=a$, is of considerable importance. It can be found from the formula

$$a_{-1} = \lim_{z \to a} \frac{1}{(n-1)!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left\{ (z-a)^n f(z) \right\} \qquad (10)$$

where n is the order of the pole. For simple poles the calculation of the residue is of particular simplicity since it reduces to

$$a_{-1} = \lim_{z \to a} (z-a) f(z) \qquad (11)$$

Example 8: If $f(z) = \dfrac{1}{z(z-2)^2}$, then $z=0$ is a simple pole, $z=2$ is a pole of order 2. Thus:

Residue at $z=0$ is $\lim\limits_{z \to 0} z \cdot \dfrac{1}{z(z-2)^2} = \dfrac{1}{4}$.

Residue at $z=2$ is

$$\lim_{z \to 2} \frac{\mathrm{d}}{\mathrm{d}z} \left\{ (z-2)^2 \cdot \frac{1}{z(z-2)^2} \right\} = -\frac{1}{4}.$$

RESIDUE THEOREM

If $f(z)$ is analytic in a region ***R*** except for a pole of order n at $z=a$ and if C is any simple closed curve in ***R*** containing $z=a$, then $f(z)$ has the form (9) (Laurent series). Integrating (9), and using the fact that

$$\oint_C \frac{\mathrm{d}z}{(z-a)^n} = \begin{cases} 2\pi i \text{ if } n=1 \\ 0 \quad \text{if } n=2,3,4,\ldots \end{cases} \tag{12}$$

(see Problem 1.7), it follows that

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i\, a_{-1} \tag{13}$$

i.e. the integral of $f(z)$ around a closed path enclosing a single pole of $f(z)$ is $2\pi i$ times the residue at the pole.

More generally, we have the following

Theorem 1: If $f(z)$ is analytic within and on the boundary C of a region ***R*** except at a finite number of poles $a, b, c, \ldots$ within ***R*****,** having residues $a_{-1}, b_{-1}, c_{-1}, \ldots$ respectively, then

$$\oint_C f(z)\,\mathrm{d}z = 2\pi i\,(a_{-1}+b_{-1}+c_{-1}+\cdots) \tag{14}$$

Cauchy's theorem and integral formula are special cases of this result, which we call the ***residue theorem***.

EVALUATION OF DEFINITE INTEGRALS

The evaluation of various real definite integrals can often be achieved by using the residue theorem together with a suitable function $f(z)$ and a suitable path or ***contour*** C, the choice of which may require great ingenuity. The following types are most common in practice.

1. $\int_0^\infty F(x)\,\mathrm{d}x$, $F(x)$ is an even function.

   Consider $\oint_C F(z)\,\mathrm{d}z$ along a contour C consisting of the line along the x axis from $-R$ to $+R$ and the semi-circle above the x axis having this line as diameter. Then let $R \to \infty$.

2. $\int_0^{2\pi} G(\sin\theta, \cos\theta)\,\mathrm{d}\theta$, G is a rational function of $\sin\theta$ and $\cos\theta$.

   Let $z = e^{i\theta}$. Then $\sin\theta = \dfrac{z - z^{-1}}{2i}$, $\cos\theta = \dfrac{z + z^{-1}}{2}$ and $\mathrm{d}z = ie^{i\theta}\mathrm{d}\theta$ or $\mathrm{d}\theta = \mathrm{d}z / iz$. The integral is equivalent to $\oint_C F(z)\,\mathrm{d}z$ where C is the unit circle with center at the origin.

3. $\int_{-\infty}^{\infty} F(x)\begin{Bmatrix}\cos(mx)\\ \sin(mx)\end{Bmatrix}dx$, $F(x)$ is a rational function.

Here we consider $\int_C F(z)\,e^{imz}dz$ where C is the same contour as that in Type 1.

4. Miscellaneous integrals involving particular contours. See Problems.

Useful property of contour integrals:

If $|f(z)| \le \dfrac{M}{R^k}$ for $z = R\,e^{i\theta}$, where $k > 1$ and M are constants, then $\lim\limits_{R\to\infty}\int_\Gamma f(z)dz = 0$ where Γ is the semi-circular arc of radius R above the x axis.

Proof:

By the result at the bottom of page 7, we have

$$\left|\int_\Gamma f(z)dz\right| \le \int_\Gamma |f(z)||dz| \le \frac{M}{R^k}\pi R = \frac{\pi M}{R^{k-1}}$$

since the length of arc $L = \pi R$. Then $\lim\limits_{R\to\infty}\left|\int_\Gamma f(z)dz\right| = 0$ and so $\lim\limits_{R\to\infty}\int_\Gamma f(z)dz = 0$.