PHY2023 Supplement 4: Derivation of the Fermi-Dirac distribution using the Lagrange Undetermined Multipliers method (non-examinable).

We require to maximise

$$\ln\Omega = \ln\left[\prod_{i=1}^{\infty}\frac{\omega_i!}{(\omega_i - n_i)!n_i!}\right] = \sum_{i=1}^{\infty}\ln\left[\frac{\omega_i!}{(\omega_i - n_i)!n_i!}\right]$$

subject to the constraints $\sum_{i=0}^{\infty} n_i = N \quad \sum_{i=0}^{\infty} n_i\, E_i = U$

Since both n_i and ω_i are, in general, large we can apply Stirling's approximation:

$$\ln\Omega \cong \sum_{i=1}^{\infty}\left(\omega_i \ln\omega_i - \omega_i - (\omega_i - n_i)\ln(\omega_i - n_i) + (\omega_i - n_i) - n_i \ln n_i + n_i\right)$$

$$\cong \sum_{i=1}^{\infty}\left(\omega_i \ln\omega_i - (\omega_i - n_i)\ln(\omega_i - n_i) - n_i \ln n_i\right)$$

Using the Lagrange Undetermined Multipliers method, we seek to maximise

$$\Gamma = \sum_{i=1}^{\infty}\left(\omega_i \ln\omega_i - (\omega_i - n_i)\ln(\omega_i - n_i) - n_i \ln n_i\right) - \lambda\sum_{i=1}^{\infty} n_i - \beta\sum_{i=1}^{\infty} n_i E_i$$

(where we have noted implicitly that N and U are constants). As before, this requires

$$\frac{\partial}{\partial n_i}\left(\omega_i \ln\omega_i - (\omega_i - n_i)\ln(\omega_i - n_i) - n_i \ln n_i - \lambda n_i - \beta n_i E_i\right) = 0 \quad \forall n_i \text{ hence}$$

$$\frac{\omega_i}{(\omega_i - n_i)} + \ln(\omega_i - n_i) - \frac{n_i}{(\omega_i - n_i)} - 1 - \ln n_i - \lambda - \beta E_i = 0 \quad \forall n_i$$

$$\therefore \ln\left(\frac{\omega_i - n_i}{n_i}\right) - \lambda - \beta E_i = 0 \quad \forall n_i$$

$$\therefore \frac{\omega_i - n_i}{n_i} = \exp(\lambda + \beta E)$$

$$\therefore \frac{n_i}{\omega_i} = \frac{1}{\exp(\lambda + \beta E) + 1}$$

As before λ and β remain to be determined using the constraint equations. However, motivated by our desire to recover the Boltzmann distribution (the analogous result for distinguishable particles not subject to the Pauli Exclusion Principle) in the limit that $n_i/\omega_i << 1$ we assign $\beta = 1/k_{\mathrm{B}}T$. (Note that $n_i/\omega_i << 1$ implies that $\exp(\lambda+\beta E) >> 1$ hence yielding the Boltzmann distribution). Motivated by our physical model of a cold Fermionic gas, we further assign $\lambda = -\beta E_F$ hence yielding the desired result

$$\boxed{\frac{n_i}{\omega_i} = \frac{1}{\exp\left((E - E_F)/k_{\mathrm{B}}T\right) + 1}}$$