PHY2023 Supplement 3: Stirling's approximation

The quantity $n!$ arises very often in statistical mechanics, because it is fundamentally involved in the calculation of the number of permutations and combinations of n objects. However, it is a function that offers very little scope for algebraic simplification. Fortunately, for realistic systems n is a large number (typically it is the umber of atoms in an ensemble and 1 gram of gas will contain of order $n = 10^{23}$ atoms). Hence the following approximation is extremely useful

For large *n:*

$$\boxed{\ln n! \approx n \ln n - n}$$

Stirling's approximation

The approximation becomes quite accurate very quickly :-

n	$n!$	$\ln n!$	$n\ln n - n$
1	1	0	-1
2	2	0.693147	-0.613706
5	120	4.787492	3.04719
10	3628800	15.10441	13.02585
20	2.43E+18	42.33562	39.91465
30	2.65E+32	74.65824	72.03592
50	3.04E+64	148.4778	145.6012
100	9.3E+157	363.7394	360.517

i.e. it is 1% accurate with $n = 100$, so the accuracy with $n = 10^{23}$ will be extremely high.

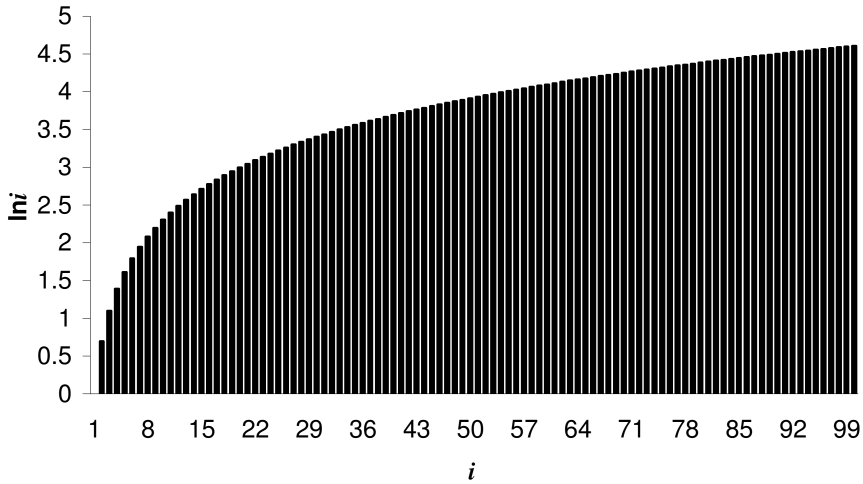
A formal proof of Stirling's approximation is beyond our scope, but a less rigorous proof goes as follows:

$$n! = \prod_{i=1}^{n} i$$

$$\therefore \ln n! = \ln\left(\prod_{i=1}^{n} i\right) = \sum_{i=1}^{n} \ln i$$

For large n, the discrete sum above can be approximated by a continuous integral

The shaded area equals the value of the discrete sum



$$\sum_{i=1}^{n} \ln i \cong \int_{1}^{n} \ln i \mathrm{d}i$$

Consider the indefinite integral $\int \ln x \mathrm{d}x$. By inspection, we see that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x \ln x - x\right) = \frac{x}{x} + \ln x - 1 = \ln x$$

$$\therefore \int \ln x \mathrm{d}x = x \ln x - x + c$$

Hence $\int_{1}^{n} \ln i \mathrm{d}i = n \ln n - n + 1$

For large n the 1 is insignificant, hence we obtain

$$\boxed{\ln n! \approx n \ln n - n}$$