AC Theory and Feedback

Complex Signals

If the signal input to a *linear* system is $V_{\mathrm{in}}\sin(\omega t)$, where ω is the (angular) frequency, then the output will be

$$V_{\mathrm{out}}(t) = V_{\mathrm{in}} G \sin(\omega t + \phi) \tag{3.1}$$

where the gain G and the **phase shift** ϕ are both functions of ω. If the input to the same system is $V_{\mathrm{in}}\cos(\omega t)$ then

$$V'_{\mathrm{out}}(t) = V_{\mathrm{in}} G \cos(\omega t + \phi). \tag{3.2}$$

Using complex numbers both equations 3.1 and 3.2 can be simultaneously expressed as

$$\mathbf{V}_{\mathrm{out}}(t) = V_{\mathrm{in}}\mathbf{G}\exp(j\omega t) \quad \text{where} \quad \mathbf{G} = G\exp(j\phi) \tag{3.3}$$

when the input is $V_{\mathrm{in}}\exp(j\omega t)$.

Complex Impedances

Ideal components are defined by their voltage-current relationships. The current I that flows when a voltage V is applied is

$$V = IR \qquad I = C\frac{\mathrm{d}V}{\mathrm{d}t} \qquad V = L\frac{\mathrm{d}I}{\mathrm{d}t} \tag{3.4a-c}$$

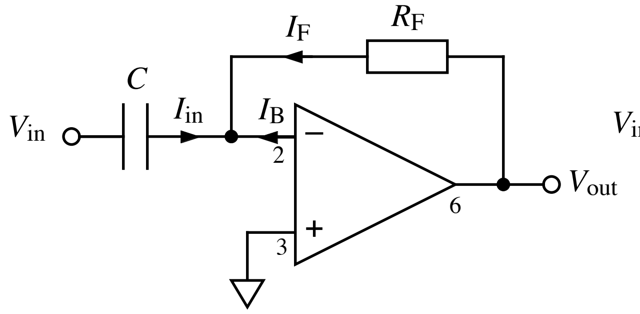
for a resistor, capacitor and inductor respectively. The **complex impedance Z** of an element at frequency ω is defined by

$$\mathbf{V} = \mathbf{IZ} \qquad \text{where} \qquad \mathbf{V} = V_0\exp(j\omega t) \qquad \text{and} \qquad \mathbf{I} = I_0\exp(j\omega t) \tag{3.5a-c}$$

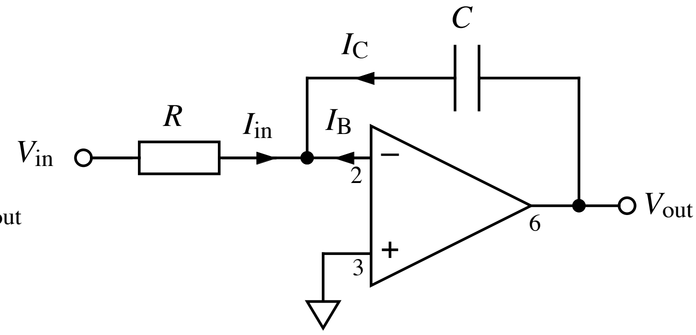
and hence the complex impedances of the ideal components are

$$\mathbf{Z}_R = R \qquad \mathbf{Z}_C = \frac{1}{j\omega C} \qquad \mathbf{Z}_L = j\omega L. \tag{3.6a-c}$$

So, we have two methods for analysing circuits involving capacitors and inductors. The **time-domain** behaviour is found by relating all the currents and voltages in the circuit using differential equations (*e.g.* equations 3.4a-c) and solving these by integrating forwards in time starting from steady state at $t = 0$. However, if we are interested in the **frequency-domain** behaviour of the circuit, the capacitors and inductors can be represented by their complex impedances and the problem can then be solved as easily as if it were a resistor network, by using complex versions of



Circuit 3.1 Differentiator



Circuit 3.2 Integrator

Kirchoff's and Ohm's laws. (The two methods give mathematically equivalent answers connected together by integral transforms similar to Fourier transforms).

The Integrators and Differentiators

Circuit 3.1 is analysed in the time domain as follows:

$$V_- = V_+ \quad \Rightarrow \quad V_2 = V_3 = 0\text{V} \tag{3.7}$$

$$I_{in} = C\frac{\mathrm{d}}{\mathrm{d}t}\left(V_{in} - V_2\right) = C\frac{\mathrm{d}V_{in}}{\mathrm{d}t} \tag{3.8}$$

$$I_B = 0 \quad \text{and} \quad I_{in} + I_F + I_B = 0 \quad \Rightarrow \quad I_F = -I_{in} \tag{3.9}$$

$$V_{out} = V_2 + I_F R_F = -CR_F\frac{\mathrm{d}V_{in}}{\mathrm{d}t}. \tag{3.10}$$

The frequency domain analysis is identical to the treatment of circuit 1.2 on Worksheet 1, but this time the impedance of the input resistor R_A is replaced by the complex impedance of the capacitor C. Therefore

$$\mathbf{V}_{out} = -\mathbf{V}_{in}R_F/Z_C = -j\omega CR_F\mathbf{V}_{in}. \tag{3.11}$$

where the factor of $-j = \exp(-j\pi/2)$ indicates that there is a $-\pi/2$ phase difference between the input and output signals.

Required Reading

Amplification – Storey (1998) §3.1–3.9 pp. 54–88 / (2006) §3.10 pp. 49–84.
Positive Feedback – Storey (1998) §4.7 pp. 145–153 / (2006) §11.1–11.2 pp. 350–356.

Exercise 3.1 Take (a) the real, and (b) the imaginary parts of both sides of equation 3.3 and hence show that it is mathematically equivalent to the pair of equations 3.1 and 3.2 .

Exercise 3.2 Show that if equations 3.5 and 3.6 are used as definitions then the equations 3.4 are satisfied.

Exercise 3.3 Analyse (*i.e.* obtain expressions relating the output to the input) circuit 3.2 in (a) the time-domain, and (b) the frequency domain.

Answer (a) $V_{\mathrm{out}}(t) = V_{\mathrm{out}}(0) - \dfrac{1}{CR}\displaystyle\int_0^t V_{\mathrm{in}}(t)\,\mathrm{d}t$ (b) $\mathbf{V}_{\mathrm{out}} = \dfrac{j\mathbf{V}_{\mathrm{in}}}{\omega CR}$

Exercise 3.4 Circuit 3.2 has been built using a 1MΩ resistor, a 4.7μF capacitor and an ideal op-amp. The output is initially constant at zero volts before a signal is applied to the input. Calculate the output if this signal is (a) a sine wave of amplitude 1.7 V and frequency 30 Hz, and (b) a pulse of height 1 V and duration 5 s.

Answer (a) $(\cos(\omega t) - 1)1.92$ mV (b) Changes linearly with time from 0 V to –1.06 V
